5.1 The Parabola

In this section you will learn how to draw the graph of the quadratic function defined by the equation
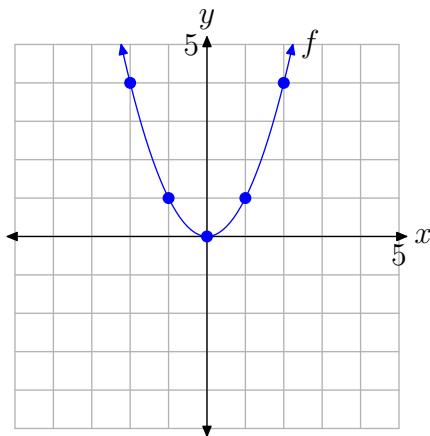
$$f(x) = a(x - h)^2 + k. \tag{1}$$

You will quickly learn that the graph of the quadratic function is shaped like a "U" and is called a *parabola*. The form of the quadratic function in **equation (1)** is called *vertex form*, so named because the form easily reveals the *vertex* or "turning point" of the parabola. Each of the constants in the vertex form of the quadratic function plays a role. As you will soon see, the constant a controls the scaling (stretching or compressing of the parabola), the constant h controls a horizontal shift and placement of the *axis of symmetry*, and the constant k controls the vertical shift.

Let's begin by looking at the *scaling* of the quadratic.

Scaling the Quadratic

The graph of the basic quadratic function $f(x) = x^2$ shown in **Figure 1**(a) is called a *parabola*. We say that the parabola in **Figure 1**(a) "opens upward." The point at $(0, 0)$, the "turning point" of the parabola, is called the *vertex* of the parabola. We've tabulated a few points for reference in the table in **Figure 1**(b) and then superimposed these points on the graph of $f(x) = x^2$ in **Figure 1**(a).

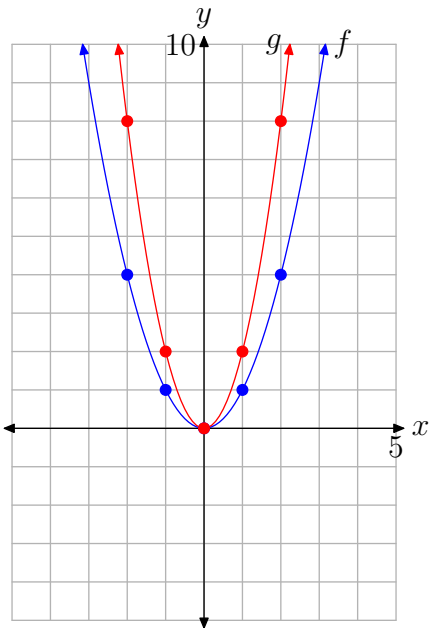| x	$f(x) = x^2$
-2	4
-1	1
0	0
1	1
2	4

(a) A basic parabola.

(b) Table of x-values and function values satisfying $f(x) = x^2$.

Figure 1.   The graph of the basic parabola is a fundamental starting point.

Now that we know the basic shape of the parabola determined by $f(x) = x^2$, let's see what happens when we scale the graph of $f(x) = x^2$ in the vertical direction. For

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[1] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

example, let's investigate the graph of $g(x) = 2x^2$. The factor of 2 has a doubling effect. Note that each of the function values of g is twice the corresponding function value of f in the table in **Figure 2**(b).

x	$f(x) = x^2$	$g(x) = 2x^2$
-2	4	8
-1	1	2
0	0	0
1	1	2
2	4	8

(a) The graphs of f and g.   (b) Table of x-values and function values
                                       satisfying $f(x) = x^2$ and $g(x) = 2x^2$.

**Figure 2.**   A stretch by a factor of 2 in the vertical direction.

When the points in the table in **Figure 2**(b) are added to the coordinate system in **Figure 2**(a), the resulting graph of g is stretched by a factor of two in the vertical direction. It's as if we had put the original graph of f on a sheet of rubber graph paper, grabbed the top and bottom edges of the sheet, and then pulled each edge in the vertical direction to stretch the graph of f by a factor of two. Consequently, the graph of $g(x) = 2x^2$ appears somewhat narrower in appearance, as seen in comparison to the graph of $f(x) = x^2$ in **Figure 2**(a). Note, however, that the vertex at the origin is unaffected by this scaling.

In like manner, to draw the graph of $h(x) = 3x^2$, take the graph of $f(x) = x^2$ and *stretch* the graph by a factor of three, tripling the y-value of each point on the original graph of f. This idea leads to the following result.

Property 2.   If a is a constant larger than 1, that is, if $a > 1$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is *stretched* by a factor of a.

▶ **Example 3.**   *Compare the graphs of* $y = x^2$, $y = 2x^2$, *and* $y = 3x^2$ *on your graphing calculator.*

Load the functions $y = x^2$, $y = 2x^2$, and $y = 3x^2$ into the Y= menu, as shown in **Figure 3**(a). Push the ZOOM button and select 6:ZStandard to produce the image shown in **Figure 3**(b).


(a)   (b)
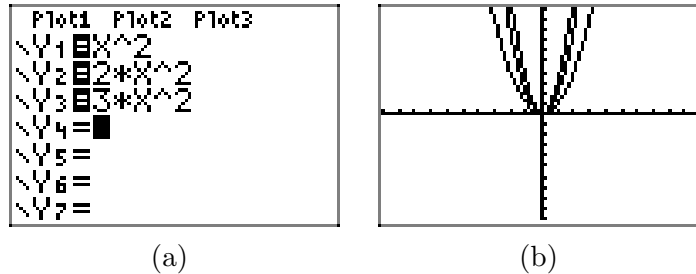
Figure 3.   Drawing $y = x^2$, $y = 2x^2$, and $y = 3x^2$ on the graphing calculator.

Note that as the "a" in $y = ax^2$ increases from 1 to 2 to 3, the graph of $y = ax^2$ stretches further and becomes, in a sense, narrower in appearance.

Next, let's consider what happens when we scale by a number that is smaller than 1 (but greater than zero — we'll deal with the negative in a moment). For example, let's investigate the graph of $g(x) = (1/2)x^2$. The factor 1/2 has a halving effect. Note that each of the function values of g is half the corresponding function value of f in the table in **Figure 4**(b).
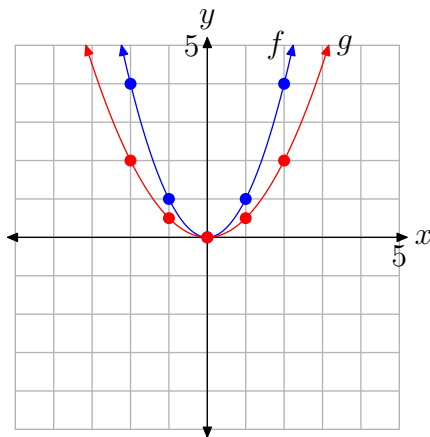
x	$f(x) = x^2$	$g(x) = (1/2)x^2$
-2	4	2
-1	1	1/2
0	0	0
1	1	1/2
2	4	2

(a) The graphs of f and g.

(a) Table of x-values and function values satisfying $f(x) = x^2$ and $g(x) = (1/2)x^2$.

Figure 4.   A compression by a factor of 2 in the vertical direction.

When the points in the table in **Figure 4**(b) are added to the coordinate system in **Figure 4**(a), the resulting graph of g is compressed by a factor of 2 in the vertical direction. It's as if we again placed the graph of $f(x) = x^2$ on a sheet of rubber graph

paper, grabbed the top and bottom of the sheet, and then *squeezed* them together by a factor of two. Consequently, the graph of $g(x) = (1/2)x^2$ appears somewhat wider in appearance, as seen in comparison to the graph of $f(x) = x^2$ in **Figure 4**(a). Note again that the vertex at the origin is unaffected by this scaling.

> **Property 4.**   If a is a constant smaller than 1 (but larger than zero), that is, if $0 < a < 1$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is *compressed* by a factor of $1/a$.

Some find **Property 4** somewhat counterintuitive. However, if you compare the function $g(x) = (1/2)x^2$ with the general form $g(x) = ax^2$, you see that $a = 1/2$. Property **4** states that the graph will be compressed by a factor of $1/a$. In this case, $a = 1/2$ and

$$\frac{1}{a} = \frac{1}{1/2} = 2.$$

Thus, **Property 4** states that the graph of $g(x) = (1/2)x^2$ should be compressed by a factor of $1/(1/2)$ or 2, which is seen to be the case in **Figure 4**(a).

▶ **Example 5.**   *Compare the graphs of $y = x^2$, $y = (1/2)x^2$, and $y = (1/3)x^2$ on your graphing calculator.*

Load the equations $y = x^2$, $y = (1/2)x^2$, and $y = (1/3)x^2$ into the Y=, as shown in **Figure 5**(a). Push the ZOOM button and select 6:ZStandard to produce the image shown in **Figure 5**(b).
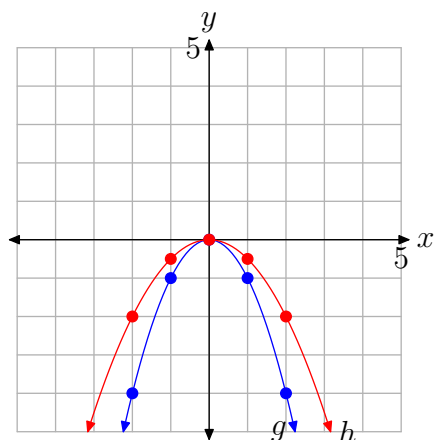

(a)   (b)
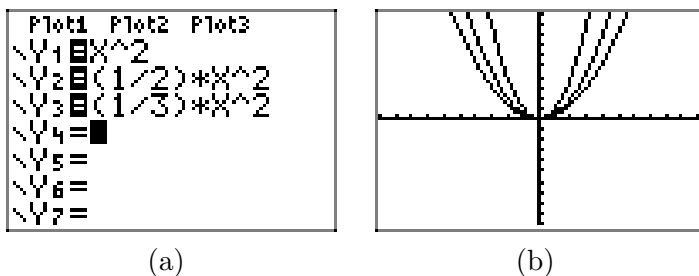
Figure 5.   Drawing $y = x^2$, $y = (1/2)x^2$, and $y = (1/3)x^2$ on the graphing calculator.

Note that as the "a" in $y = ax^2$ decreases from 1 to 1/2 to 1/3, the graph of $y = ax^2$ compresses further and becomes, in a sense, wider in appearance.

Vertical Reflections

Let's consider the graph of $g(x) = ax^2$, when $a < 0$. For example, consider the graphs of $g(x) = -x^2$ and $h(x) = (-1/2)x^2$ in **Figure 6**.

x	$g(x) = -x^2$	$h(x) = (-1/2)x^2$
-2	-4	-2
-1	-1	$-1/2$
0	0	0
1	-1	$-1/2$
2	-4	-2

(a) The graphs of g and h.

(b) Table of x-values and function values satisfying $g(x) = -x^2$ and $h(x) = (-1/2)x^2$.

Figure 6.   A vertical reflection across the x-axis.

When the table in **Figure 6**(b) is compared with the table in **Figure 4**(b), it is easy to see that the numbers in the last two columns are the same, but they've been negated. The result is easy to see in **Figure 6**(a). The graphs have been reflected across the x-axis. Each of the parabolas now "opens downward."

However, it is encouraging to see that the scaling role of the constant a in $g(x) = ax^2$ has not changed. In the case of $h(x) = (-1/2)x^2$, the y-values are still "compressed" by a factor of two, but the minus sign negates these values, causing the graph to reflect across the x-axis. Thus, for example, one would think that the graph of $y = -2x^2$ would be *stretched* by a factor of two, then reflected across the x-axis. Indeed, this is correct, and this discussion leads to the following property.

> **Property 6.**   If $-1 < a < 0$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is compressed by a factor of $1/|a|$, then reflected across the x-axis. Secondly, if $a < -1$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is stretched by a factor of $|a|$, then reflected across the x-axis.

Again, some find **Property 6** confusing. However, if you compare $g(x) = (-1/2)x^2$ with the general form $g(x) = ax^2$, you see that $a = -1/2$. Note that in this case, $-1 < a < 0$. **Property 6** states that the graph will be compressed by a factor of $1/|a|$. In this case, $a = -1/2$ and

$$\frac{1}{|a|} = \frac{1}{|-1/2|} = 2.$$

That is, **Property 6** states that the graph of $g(x) = (-1/2)x^2$ is compressed by a factor of $1/(|-1/2|)$, or 2, then reflected across the x-axis, which is seen to be the case in **Figure 6**(a). Note again that the vertex at the origin is unaffected by this scaling and reflection.

▶ **Example 7.** *Sketch the graphs of $y = -2x^2$, $y = -x^2$, and $y = (-1/2)x^2$ on your graphing calculator.*

Each of the equations were loaded separately into `Y1` in the `Y=` menu. In each of the images in **Figure 7**, we selected `6:ZStandard` from the `ZOOM` menu to produce the image.

(a) $y = -2x^2$ (b) $y = -x^2$ (c) $y = (-1/2)x^2$
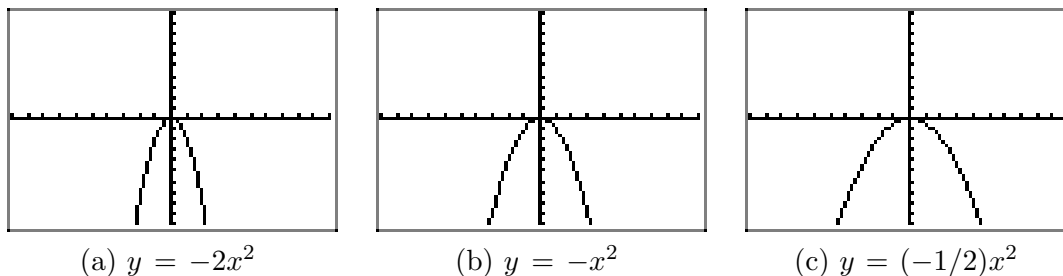
Figure 7.

In **Figure 7**(b), the graph of $y = -x^2$ is a reflection of the graph of $y = x^2$ across the x-axis and opens downward. In **Figure 7**(a), note that the graph of $y = -2x^2$ is stretched vertically by a factor of 2 (compare with the graph of $y = -x^2$ in **Figure 7**(b)) and reflected across the x-axis to open downward. In **Figure 7**(c), the graph of $(-1/2)x^2$ is compressed by a factor of 2, appears a bit wider, and is reflected across the x-axis to open downward.

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Horizontal Translations

The graph of $g(x) = (x + 1)^2$ in **Figure 8**(a) shows a basic parabola that is shifted one unit to the left. Examine the table in **Figure 8**(b) and note that the equation $g(x) = (x + 1)^2$ produces the same y-values as does the equation $f(x) = x^2$, the only difference being that these y-values are calculated at x-values that are one unit less than those used for $f(x) = x^2$. Consequently, the graph of $g(x) = (x + 1)^2$ must shift one unit to the left of the graph of $f(x) = x^2$, as is evidenced in **Figure 8**(a).

Note that this result is counterintuitive. One would think that replacing x with $x + 1$ would shift the graph one unit to the right, but the shift actually occurs in the opposite direction.

Finally, note that this time the vertex of the parabola has shifted 1 unit to the left and is now located at the point $(-1, 0)$.

We are led to the following conclusion.

Property 8. If $c > 0$, then the graph of $g(x) = (x + c)^2$ is shifted c units to the left of the graph of $f(x) = x^2$.
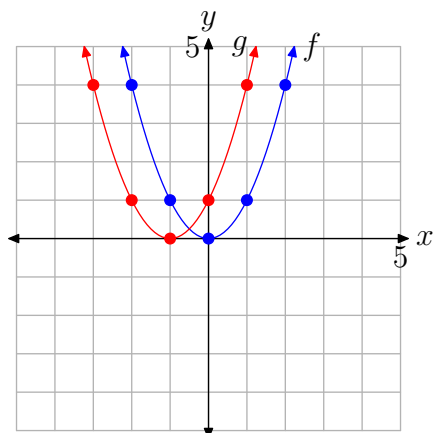
(a) The graphs of f and g.

x	$f(x) = x^2$	x	$g(x) = (x+1)^2$
-2	4	-3	4
-1	1	-2	1
0	0	-1	0
1	1	0	1
2	4	1	4

(a) Table of x-values and function values satisfying $f(x) = x^2$ and $g(x) = (x+1)^2$.

Figure 8.   A horizontal shift or translation.

A similar thing happens when you replace x with $x - 1$, only this time the graph is shifted one unit to the right.

▶ **Example 9.**   *Sketch the graphs of* $y = x^2$ *and* $y = (x - 1)^2$ *on your graphing calculator.*

Load the equations $y = x^2$ and $y = (x - 1)^2$ into the Y= menu, as shown in **Figure 9**(a). Push the ZOOM button and select 6:ZStandard to produce the image shown in **Figure 9**(b).
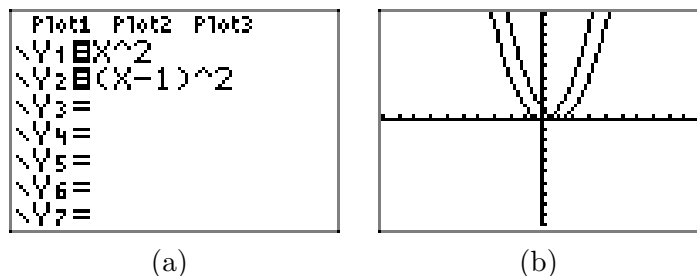
(a)

(b)

Figure 9.   Drawing $y = x^2$ and $y = (x-1)^2$ on the graphing calculator.

Note that the graph of $y = (x - 1)^2$ is shifted 1 unit to the right of the graph of $y = x^2$ and the vertex of the graph of $y = (x - 1)^2$ is now located at the point $(1, 0)$.
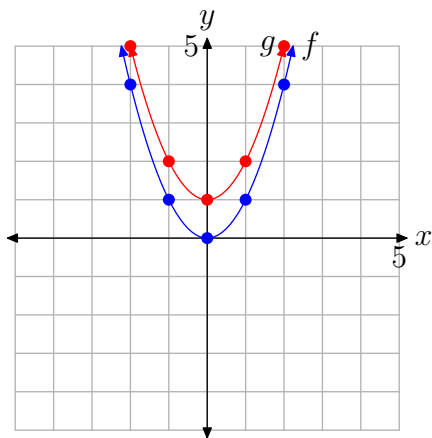
We are led to the following property.

Property 10.   If $c > 0$, then the graph of $g(x) = (x - c)^2$ is shifted c units to the right of the graph of $f(x) = x^2$.

Vertical Translations

The graph of $g(x) = x^2 + 1$ in **Figure 10**(a) is shifted one unit upward from the graph of $f(x) = x^2$. This is easy to see as both equations use the same x-values in the table in **Figure 10**(b), but the function values of $g(x) = x^2 + 1$ are one unit larger than the corresponding function values of $f(x) = x^2$.

Note that the vertex of the graph of $g(x) = x^2 + 1$ has also shifted upward 1 unit and is now located at the point $(0, 1)$.

x	$f(x) = x^2$	$g(x) = x^2 + 1$
-2	4	5
-1	1	2
0	0	1
1	1	2
2	4	5

Figure 10.   A vertical shift or translation.

The above discussion leads to the following property.

Property 11.   If $c > 0$, the graph of $g(x) = x^2 + c$ is shifted c units upward from the graph of $f(x) = x^2$.

In a similar vein, the graph of $y = x^2 - 1$ is shifted downward one unit from the graph of $y = x^2$.

▶ **Example 12.**   *Sketch the graphs of $y = x^2$ and $y = x^2 - 1$ on your graphing calculator.*

Load the equations $y = x^2$ and $y = x^2 - 1$ into the Y= menu, as shown in **Figure 11**(a). Push the ZOOM button and select 6:ZStandard to produce the image shown in **Figure 11**(b).

Note that the graph of $y = x^2 - 1$ is shifted 1 unit downward from the graph of $y = x^2$ and the vertex of the graph of $y = x^2 - 1$ is now at the point $(0, -1)$.

(a)   (b)
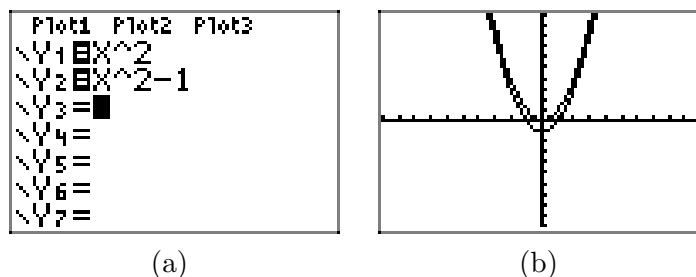
Figure 11.   Drawing $y = x^2$ and $y = x^2 - 1$ on the graphing calculator.

The above discussion leads to the following property.

> **Property 13.**   If $c > 0$, the graph of $g(x) = x^2 - c$ is shifted c units downward from the graph of $f(x) = x^2$.

The Axis of Symmetry

In **Figure 1**, the graph of $y = x^2$ is symmetric with respect to the y-axis. One half of the parabola is a mirror image of the other with respect to the y-axis. We say the y-axis is acting as the *axis of symmetry*.

If the parabola is reflected across the x-axis, as in **Figure 6**, the axis of symmetry doesn't change. The graph is still symmetric with respect to the y-axis. Similar comments are in order for scalings and vertical translations. However, if the graph of $y = x^2$ is shifted right or left, then the axis of symmetry will change.

▶ **Example 14.**   *Sketch the graph of $y = -(x + 2)^2 + 3$.*

Although not required, this example is much simpler if you perform reflections before translations.

> **Tip 15.**   *If at all possible, perform scalings and reflections before translations.*

In the series shown in **Figure 12**, we first perform a reflection, then a horizontal translation, followed by a vertical translation.

- In **Figure 12**(a), the graph of $y = -x^2$ is a reflection of the graph of $y = x^2$ across the x-axis and opens downward. Note that the vertex is still at the origin.

- In **Figure 12**(b), we've replaced x with $x + 2$ in the equation $y = -x^2$ to obtain the equation $y = -(x+2)^2$. The effect is to shift the graph of $y = -x^2$ in **Figure 12**(a) 2 units to the left to obtain the graph of $y = -(x+2)^2$ in **Figure 12**(b). Note that the vertex is now at the point $(-2, 0)$.

- In **Figure 12**(c), we've added 3 to the equation $y = -(x+2)^2$ to obtain the equation $y = -(x+2)^2 + 3$. The effect is to shift the graph of $y = -(x+2)^2$ in **Figure 12**(b)

upward 3 units to obtain the graph of $y = -(x+2)^2 + 3$ in **Figure 12**(c). Note that the vertex is now at the point $(-2, 3)$.

(a) $y = -x^2$ (b) $y = -(x+2)^2$ (c) $y = -(x+2)^2 + 3$
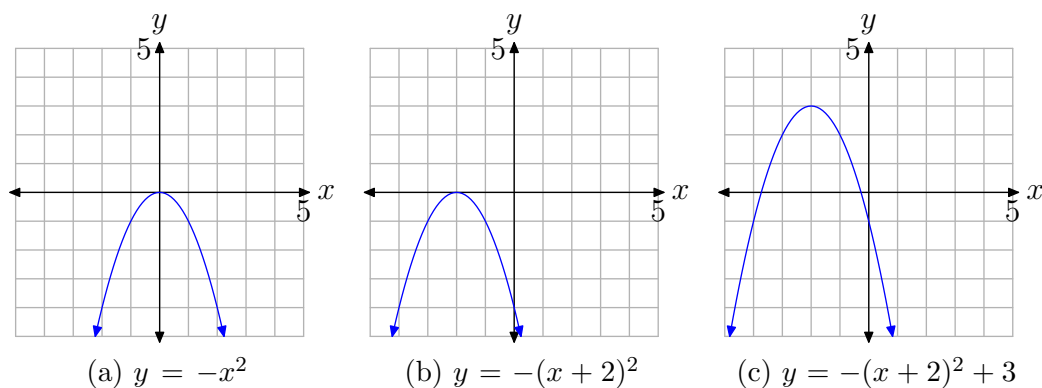
Figure 12.   Finding the graph of $y = -(x+2)^2 + 3$
through a series of transformations.

In practice, we can proceed more quickly. Analyze the equation $y = -(x+2)^2 + 3$. The minus sign tells us that the parabola "opens downward." The presence of $x+2$ indicates a shift of 2 units to the left. Finally, adding the 3 will shift the graph 3 units upward. Thus, we have a parabola that "opens downward" with vertex at $(-2, 3)$. This is shown in **Figure 13**.
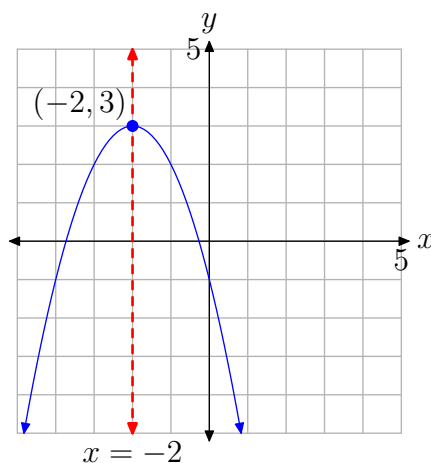
Figure 13.   The axis of symmetry
passes through the vertex.

The axis of symmetry passes through the vertex $(-2, 3)$ in **Figure 13** and has equation $x = -2$. Note that the right half of the parabola is a mirror image of its left half across this axis of symmetry. We can use the axis of symmetry to gain an accurate plot of the parabola with minimal plotting of points.

Guidelines for Using the Axis of Symmetry.

- Start by plotting the vertex and axis of symmetry as shown in **Figure 14**(a).

- Next, compute two points on either side of the axis of symmetry. We choose $x = -1$ and $x = 0$ and compute the corresponding y-values using the equation $y = -(x+2)^2 + 3$.

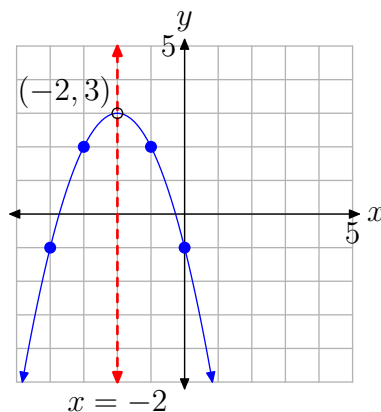| x	$y = -(x+2)^2 + 3$
-1	2
0	-1

  Plot the points from the table, as shown in **Figure 14**(b).

- Finally, plot the mirror images of these points across the axis of symmetry, as shown in **Figure 14**(c).

(a)   (b)   (c)
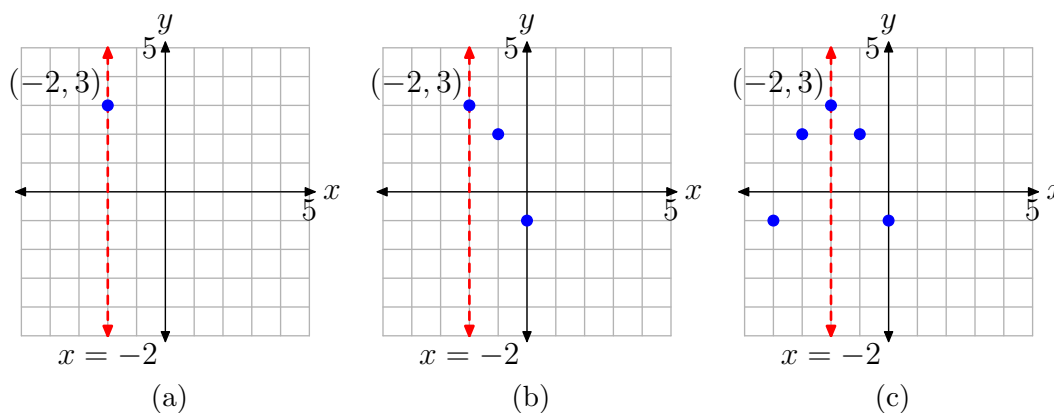
Figure 14.   Using the axis of symmetry to establish accuracy.

The image in **Figure 14**(c) clearly contains enough information to complete the graph of the parabola having equation $y = -(x+2)^2 + 3$ in **Figure 15**.

Figure 15.   An accurate plot of $y = -(x+2)^2 + 3$.

Version: Fall 2007

Let's summarize what we've seen thus far.

Summary 16.   *The form of the quadratic function*

$$f(x) = a(x - h)^2 + k$$

*is called **vertex form**. The graph of this quadratic function is a **parabola**.*

1. *The graph of the parabola opens upward if $a > 0$, downward if $a < 0$.*
2. *If the magnitude of a is larger than 1, then the graph of the parabola is stretched by a factor of a. If the magnitude of a is smaller than 1, then the graph of the parabola is compressed by a factor of $1/a$.*
3. *The parabola is translated h units to the right if $h > 0$, and h units to the left if $h < 0$.*
4. *The parabola is translated k units upward if $k > 0$, and k units downward if $k < 0$.*
5. *The coordinates of the vertex are (h, k).*
6. *The axis of symmetry is a vertical line through the vertex whose equation is $x = h$.*

Let's look at one final example.

▶ **Example 17.**   *Use the technique of **Example 14** to sketch the graph of $f(x) = 2(x - 2)^2 - 3$.*

Compare $f(x) = 2(x - 2)^2 - 3$ with $f(x) = a(x - h)^2 + k$ and note that $a = 2$. Hence, the parabola has been "stretched" by a factor of 2 and opens upward. The presence of $x - 2$ indicates a shift of 2 units to the right; and subtracting 3 shifts the parabola 3 units downward. Therefore, the vertex will be located at the point $(2, -3)$ and the axis of symmetry will be the vertical line having equation $x = 2$. This is shown in **Figure 16**(a).

Note. Some prefer a more strict comparison of $f(x) = 2(x - 2)^2 - 3$ with the general vertex form $f(x) = a(x - h)^2 + k$, yielding $a = 2$, $h = 2$, and $k = -3$. This immediately identifies the vertex at (h, k), or $(2, -3)$.

Next, evaluate the function $f(x) = 2(x - 2)^2 - 3$ at two points lying to the right of the axis of symmetry (or to the left, if you prefer). Because the axis of symmetry is the vertical line $x = 2$, we choose to evaluate the function at $x = 3$ and 4.

$$f(3) = 2(3 - 2)^2 - 3 = -1$$
$$f(4) = 2(4 - 2)^2 - 3 = 5$$

This gives us two points to the right of the axis of symmetry, $(3, -1)$ and $(4, 5)$, which we plot in **Figure 16**(b).

Finally, we plot the mirror images of $(3, -1)$ and $(4, 5)$ across the axis of symmetry, which gives us the points $(1, -1)$ and $(0, 5)$, respectively. These are plotted in **Figure 16**(c). We then draw the parabola through these points.

(a)   (b)   (c)
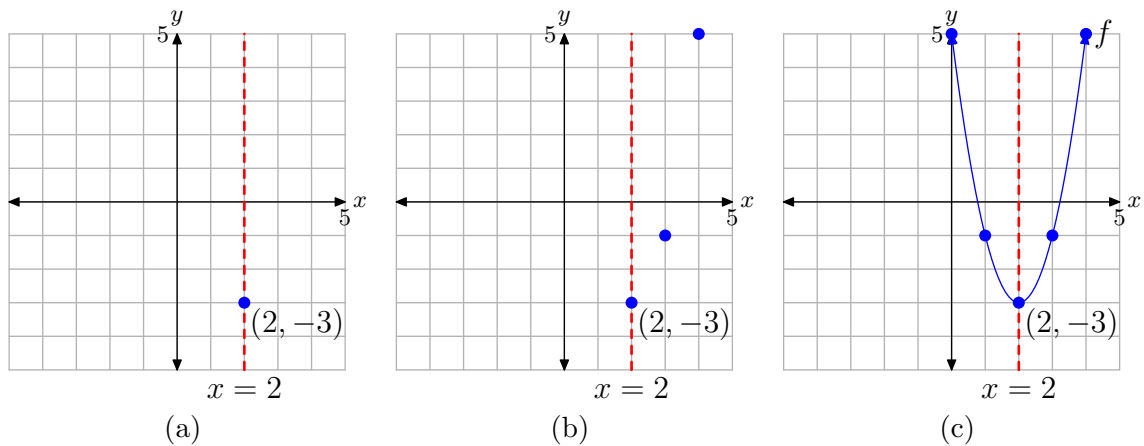
Figure 16.   Creating the graph of $f(x) = 2(x-2)^2 - 3$.

Let's finish by describing the domain and range of the function defined by the rule $f(x) = 2(x-2)^2 - 3$. If you use the intuitive notion that the domain is the set of "permissible x-values," then one can substitute any number one wants into the equation $f(x) = 2(x-2)^2 - 3$. Therefore, the domain is all real numbers, which we can write as follows: Domain $= \mathbb{R}$ or Domain $= (-\infty, \infty)$.

You can also project each point on the graph of $f(x) = 2(x-2)^2 - 3$ onto the x-axis, as shown in **Figure 17**(a). If you do this, then the entire axis will "lie in shadow," so once again, the domain is all real numbers.

(a)   (b)
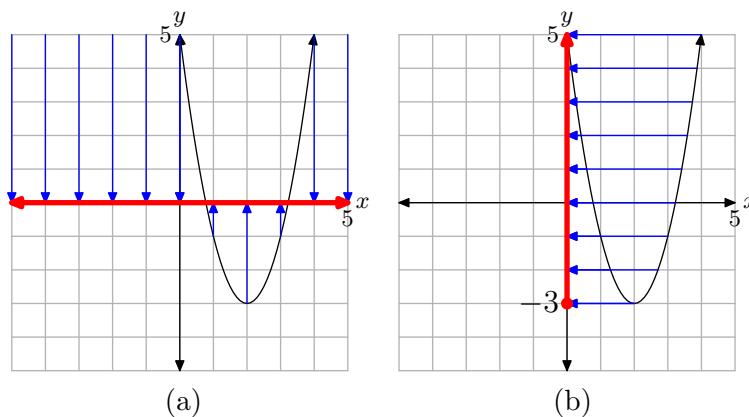
Figure 17.   Projecting to find
(a) the domain and (b) the range.

To determine the range of the function $f(x) = 2(x-2)^2 - 3$, project each point on the graph of f onto the y-axis, as shown in **Figure 17**(b). On the y-axis, all points greater than or equal to -3 "lie in shadow," so the range is described with Range $= \{y : y \geq -3\} = [-3, \infty)$.

The following summarizes how one finds the domain and range of a quadratic function that is in vertex form.

Summary 18.   *The domain of the quadratic function*

$$f(x) = a(x - h)^2 + k,$$

regardless of the values of the parameters a, h, and k, is the set of all real numbers, easily described with $\mathbb{R}$ or $(-\infty, \infty)$. On the other hand, the range depends upon the values of a and k.

- *If $a > 0$, then the parabola opens upward and has vertex at (h, k). Consequently, the range will be*

$$[k, \infty) = \{y : y \geq k\}.$$

- *If $a < 0$, then the parabola opens downward and has vertex at (h, k). Consequently, the range will be*

$$(-\infty, k] = \{y : y \leq k\}.$$